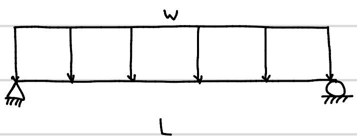
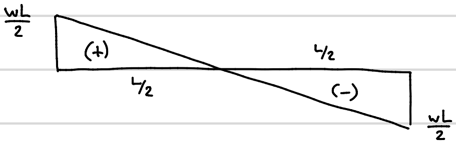
January 29, 2026
Solids Review
Part III - Beam Deflections

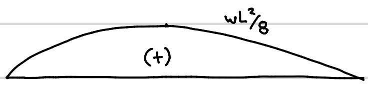


$q(x) = w$



$V(x) = \int -q(x)\,dx = \int -w\,dx = -wx + \cancel{C}$ $\quad wL/2$ $\quad @\ x=0 \quad V(x) = wL/2$

$V(x) = \frac{wL}{2} - wx$



$M(x) = \int V(x)\,dx = \int \frac{wL}{2} - wx\,dx = \frac{wLx}{2} - \frac{wx^2}{2} + \cancel{C}$

$M(x) = \frac{wLx}{2} - \frac{wx^2}{2}$

Curvature $\quad K = \frac{M(x)}{EI} \qquad K = \frac{d^2v(x)}{dx^2} = v''$

$v'' = \frac{M}{EI}$

modulus of elasticity (E) — moment of inertia (I)

$EIv'' = M(x)$ moment-curvature relation

$EIv' = \int M(x)\,dx$ $\quad v'$ rotation

$EIv = \iint M(x)\,dx$ $\quad v$ deflection

$EIv'' = M(x)$

$EIv''' = V(x)$

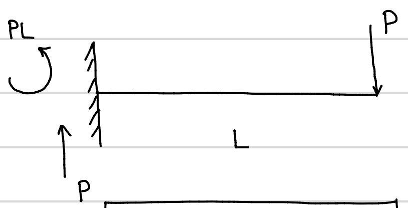
$EIv'''' = -q(x)$

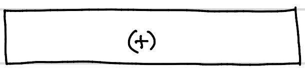
$EIv'' = -\frac{wx^2}{2} + \frac{wLx}{2}$

$EIv' = \int -\frac{wx^2}{2} + \frac{wLx}{2}\,dx = -\frac{wx^3}{6} + \frac{wLx^2}{4} + C \quad @\ x = L/2 \quad v' = 0 \quad \therefore \quad C = \frac{-wL^3}{24}$ symmetry

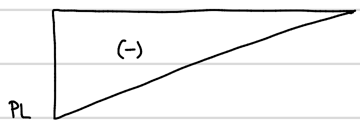
$EIv = \int -\frac{wx^3}{6} + \frac{wLx^2}{4} - \frac{wL^3}{24} = -\frac{wx^4}{24} + \frac{wLx^3}{12} - \frac{wL^3}{24}x + D \quad @\ x=0 \quad v=0 \quad \therefore \quad D = 0$ B.C.s

$EIv(x) = -\frac{wx^4}{24} + \frac{wLx^3}{12} - \frac{wL^3x}{24} \qquad v_{max}$? set $v'(x) = 0 \quad x = L/2 \qquad v(L/2) = \frac{-5wL^4}{384EI}$ *





$V(x) = P$



$M(x) = \int P\,dx = Px + C$ @ $x=0$ $M(x) = -PL$ natural B.C.

$M(x) = Px - PL$

$EIv'' = M(x)$ $EIv'' = Px - PL$

$EIv' = \int Px - PL\,dx = \frac{Px^2}{2} - PLx + \cancel{C}^{\,0}$ @ $x=0$ $v'=0$ $\therefore C = 0$ essential B.C.

$EIv = \int \frac{Px^2}{2} - PLx\,dx = \frac{Px^3}{6} - \frac{PLx^2}{2} + D$ @ $x=0$ $v=0$ $\therefore D = 0$ essential B.C.

rotation $EIv' = \frac{Px^2}{2} - PLx$ v'_{max} @ $x=L$ ($M(x)=0$ @ free end)

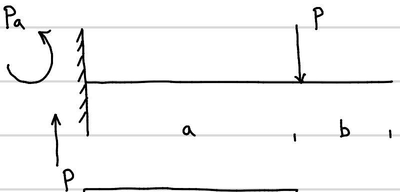
** $v'_{max} = \frac{-PL^2}{2EI}$

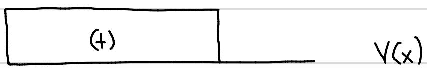
deflection $EIv = \frac{Px^3}{6} - \frac{PLx^2}{2}$ v_{max} @ $x=L$

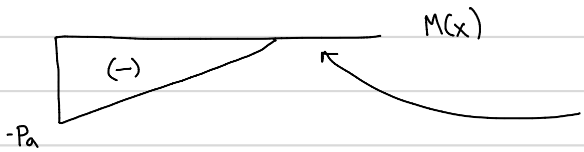
** $v_{max} = \frac{-PL^3}{3EI}$

natural B.C.s $V(x), M(x)$ shear/moment

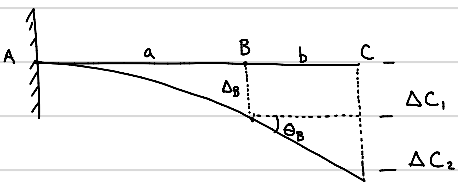
essential B.C.s $v'(x), v(x)$ rotation/deflection







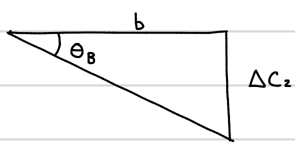
No moment = no curvature = no bending ∴ remains straight



$\Delta_{total} = \Delta C_1 + \Delta C_2$

Point B $\quad \Delta_B \equiv v_B = \dfrac{-Pa^3}{3EI} = \Delta C_1$

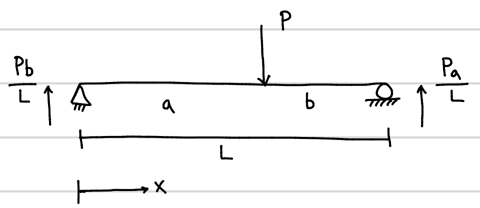
$\theta_B \equiv v'_B = \dfrac{-Pa^2}{2EI}$ (from prior derivation)

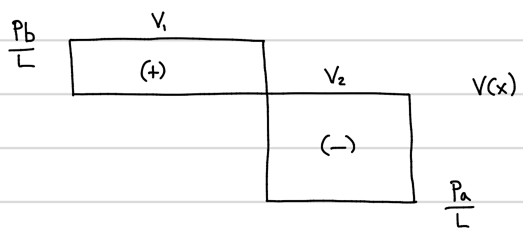


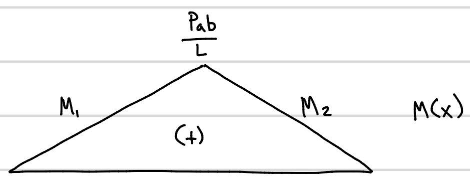
$\tan\theta_B = \dfrac{\Delta C_2}{b}$ $\quad \Delta C_2 = b \cdot \tan\theta_B$ $\quad$ for small θ_B $\quad \tan\theta_B \approx \theta_B$ (radians)

$\therefore \Delta C_2 = b \cdot \theta_B = \dfrac{-Pa^2 b}{2EI}$

$$\Delta_{total} = \frac{-Pa^3}{3EI} - \frac{Pa^2 b}{2EI} = \Delta_{max}$$







$V_1(x) \quad 0 \le x \le a \qquad V_1(x) = \frac{Pb}{L}$

$V_2(x) \quad a \le x \le L \qquad V_2(x) = -\frac{Pa}{L}$

$M_1(x) = \int \frac{Pb}{L}\,dx = \frac{Pbx}{L} + \cancel{C}^{\,0} \qquad M_1(x) = \frac{Pbx}{L} \qquad @\ x = 0 \quad M_1 = 0$

$M_2(x) = \int -\frac{Pa}{L}\,dx = -\frac{Pax}{L} + \cancel{C}^{\,Pa} \qquad M_2(x) = -\frac{Pax}{L} + Pa \qquad @\ x = L \quad M_2 = 0$

$EIv_1'(x) = \frac{Pbx^2}{2L} + C_1 \qquad EIv_2'(x) = -\frac{Pax^2}{2L} + Pax + C_2$

$EIv_1(x) = \frac{Pbx^3}{6L} + C_1x + D_1 \qquad EIv_2(x) = -\frac{Pax^3}{6L} + \frac{Pax^2}{2} + C_2x + D_2$

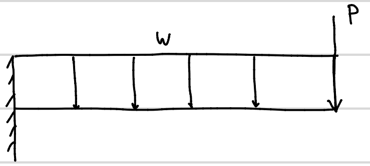
4 unknowns C_1, D_1, C_2, D_2

- Boundary Conditions $\quad v_1(0) = 0 \qquad v_2(L) = 0$
- Compatibility $\quad v_1(a) = v_2(a) \qquad v_1'(a) = v_2'(a)$
- Symmetry NOT AVAILABLE / NOT NEEDED

} B.C.s and compatibility always sufficient to solve differential equation

Principle of Superposition

* valid for linear problems

0. 

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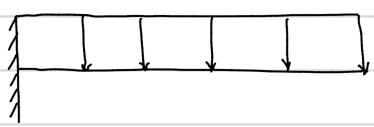
1. 

$$M_1(x) = Px - PL$$

$$v_1'(x) = \frac{1}{EI}\left(\frac{Px^2}{2} - PLx\right)$$

$$v_1(x) = \frac{1}{EI}\left(\frac{Px^3}{6} - \frac{PLx^2}{2}\right)$$

+

2. 

$$M_2(x) = wLx - \frac{wx^2}{2} - \frac{wL^2}{2}$$

$$v_2'(x) = \frac{1}{EI}\left(\frac{wLx^2}{2} - \frac{wx^3}{6} - \frac{wL^2x}{2}\right)$$

$$v_2(x) = \frac{1}{EI}\left(\frac{wLx^3}{6} - \frac{wx^4}{24} - \frac{wL^2x^2}{4}\right)$$

$$v_0' = v_1' + v_2'$$

$$v_0 = v_1 + v_2$$